

RAMAKRISHNA MISSION VIDYAMANDIRA
(Residential Autonomous College affiliated to University of Calcutta)
M.A./M.Sc. FIRST SEMESTER EXAMINATION, MARCH 2022
FIRST YEAR [BATCH 2021-23]
Mathematics

Date : 21/03/2022

Time : 11.00 am – 1.00 pm

Linear Algebra 1

Paper : MTM CC2

Full Marks:50

Answer all the questions, maximum one can score 50.

1. (a) Let V be a n dimensional vector space over the field $\mathbb{F}$. Let T be a linear map from V to V such that $T^2 = 0$. Prove that the image of T is contained in the kernel of T and hence rank of T is at most $n/2$.
- (b) Let V be a vector space over the complex number field $\mathbb{C}$. Let f and g are linear functional on V such that $h(\alpha) = f(\alpha)g(\alpha)$ is also a linear functional on V . Prove that either $f = 0$ or $g = 0$.
- (c) Let A be an $m \times n$ matrix with real entries. Prove that $A = 0$ if and only if $\text{trace}(A^t A) = 0$.
- (d) Let V be a finite dimensional vector space over the field $\mathbb{F}$. Show that $T \rightarrow T^t$ is an isomorphism of $\mathbf{L}(V, V)$ onto $\mathbf{L}(V^*, V^*)$. [4+4+4+4]
2. (a) Let V be the vector space of all continuous function from $\mathbb{R}$ into $\mathbb{R}$. Let T be a linear operator on V defined by

$$T(f(x)) = \int_0^x f(t) dt.$$

Prove that T has no characteristic values.

- (b) Prove that a nonzero 2×2 nilpotent matrix over $\mathbb{C}$ is similar to $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.
- (c) Prove that if E is the projection on R along N , then $I - E$ is the projection on N along R .
- (d) Let V be a finite dimensional vector space over the field of complex numbers $\mathbb{C}$. Let T be a linear operator on V . Prove that T is diagonalizable if and only if T is annihilated by some polynomial over $\mathbb{C}$ which has distinct roots. [4+4+4+5]
3. (a) Let V be the space of $n \times n$ matrices over the field $\mathbb{F}$, and let A be a fixed $n \times n$ matrix over $\mathbb{F}$. Let T be a linear operator on V defined by $T(B) = AB - BA$. Prove that if A is nilpotent matrix, then characteristic polynomial of T is x^{n^2} .
- (b) Let V be a finite dimensional vector space over the field of complex numbers $\mathbb{C}$. Let T be a linear operator on V and let D be the diagonalizable part of T . Prove that if $f(x) \in \mathbb{C}[x]$, then the diagonalizable part of $f(T)$ is $f(D)$.
- (c) Let V be the space of $n \times n$ matrices over the field of complex numbers $\mathbb{C}$, with the inner product

$$\langle A, B \rangle = \text{trace}(AB^*).$$

Find the orthogonal complement of the subspace of diagonal matrices.

- (d) Let V be a finite dimensional vector space over the field of complex numbers $\mathbb{C}$. Let T be a linear operator on V . Prove that T is normal if and only if $T = T_1 + iT_2$, where T_1 and T_2 are self-adjoint operators which commute. [4+4+4+4]

4. (a) Let n be a positive integer and $\mathbb{F}$ a field. Let W be the set of all vectors $(x_1, \dots, x_n) \in \mathbb{F}^n$ such that $x_1 + \dots + x_n = 0$.

(i) Prove that W^0 consists of all linear functionals f of the form

$$f(x_1, \dots, x_n) = c \sum_{j=1}^n x_j \text{ for some } c \in \mathbb{F}.$$

(ii) Show that the dual space W^* of W can be ‘naturally’ identified with the linear functionals

$$f(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

on $\mathbb{F}^n$ which satisfy $c_1 + \dots + c_n = 0$.

(b) Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and $T \in L(V, V)$. Prove that if T^2 has a cyclic vector, then T has a cyclic vector.

(c) Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and f be any bilinear form on V . Let $W = \{\beta \in V \mid f(\alpha, \beta) = 0 \text{ for all } \alpha \in V\}$.

Prove that $\text{rank } f = \dim V - \dim W$.

[4+4+3]

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